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Generalized Abel–Gončarov Bases in Spaces of Holomorphic Functions

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1. INTRODUCTION

In this paper we use functional analysis methods, namely the theory of bases in nuclear Fréchet spaces, in order to obtain results on polynomial expansions of holomorphic functions. Let $\mathscr{F}_R (0 < R \leqslant \infty)$ denote the vector space of holomorphic functions on the open disc $D_R = \{z: |z| < R\}$ with the topology of uniform convergence on compact subsets of D_R. We will consider the space $\mathscr{F}_R$ as a Köthe sequence space Λ_R by mapping a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ onto the sequence $(a_k)_{k=0}^{\infty}$. Let $(P_n)_{k=0}^{\infty}$ be a sequence of polynomials of degree $n = 0, 1, 2,...$; we will give a necessary and sufficient condition for certain sequences $(P_n)_{k=0}^{\infty}$ to be a basis in one fixed space $\mathscr{F}_R$ $(0 < R \leqslant \infty)$ (see proposition 1). In [4] and [5] Dragilev gives necessary and sufficient conditions for $(P_n)_{n=0}^{\infty}$ to be a basis in all spaces $\mathscr{F}_r$ $(r > R)$, where R is a fixed positive number.

There are two important special cases: the Gončarov polynomials and the remainder polynomials. Let $(z_k)_{k=0}^{\infty}$ be an arbitrary sequence of complex numbers, the Gončarov polynomials $G_n(z; z_0,..., z_{n-1})$ are defined by $G_0(z) \equiv 1$ and

$$G_n(z; z_0,..., z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0,..., z_{k-1}), \qquad n = 1, 2,... .$$

These polynomials are biorthogonal to the linear functionals $L_n(f) = f^{(n)}(z_n)$, $n = 0, 1, 2,...$, where $f \in \mathscr{F}_R$, i.e. $L_n(G_m) = \delta_{nm}$. The question, when the Gončarov polynomials constitute a basis in $\mathscr{F}_R$, i.e. each $f \in \mathscr{F}_R$ is uniquely representable by the series $f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_n) G_n(z; z_0,..., z_{n-1})$ (convergence in the topology of $\mathscr{F}_R$), was considered by many authors (Gončarov [9], Evgrafov [6], Dragilev [4, 12]).

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The remainder polynomials are defined by $B_0(z) \equiv 1$ and $B_n(z; z_0, \ldots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, \ldots, z_{k-1})$, $n = 1, 2, \ldots$. For a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathscr{F}_R$, let $\mathscr{S}$ denote the operator which transforms f into $\mathscr{S}f(z) = \sum_{k=1}^{\infty} a_k z^{k-1}$, and let $\mathscr{S}^k$ be defined as the kth successive iterate of $\mathscr{S}$. The remainder polynolials are biorthogonal to the linear functionals

$$l_n(f) = \mathscr{S}^n f(z_n) \qquad \text{for} \quad f \in \mathscr{F}_R$$

and we have the problem of the unique expansion

$$f(z) = \sum_{n=0}^{\infty} \mathscr{S}^n f(z_n)\, B_n(z; z_0, \ldots, z_{n-1})$$

(see Pommiez [16], Dragilev and Čuhlova [5], Buckholtz and Frank [3]).

Let $(d_k)_{k=1}^{\infty}$ be a nondecreasing sequence of positive numbers and define the so called Gel'fond-Leont'ev [8] derivative of a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathscr{F}_R$ by

$$\mathscr{D}f(z) = \sum_{k=1}^{\infty} d_k a_k z^{k-1}.$$

This is a generalization of the orinary derivative, where $d_k = k$ $(k = 1, 2, \ldots)$, and of $\mathscr{S}$, where $d_k \equiv 1$.

Denote by $Q_n(z; z_0, \ldots, z_{n-1})$ the corresponding polynomials (see Section 2.) biorthogonal to the linear functionals

$$\mathscr{L}_n(f) = \mathscr{D}^n f(z_n).$$

In proposition 1 we characterize those sequences $(z_k)_{k=0}^{\infty}$ and $(d_k)_{k=1}^{\infty}$, for which the corresponding polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ constitute a basis in $\mathscr{F}_R$. With the help of proposition 1 it is possible to show that a basis of polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ in $\mathscr{F}_R$ is also a basis in any space $\mathscr{F}_{R'}$ $(R' > R)$. For a given basis of polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ in $\mathscr{F}_{R^*}$ $(0 < R^* < \infty)$ it is now possible to ask for the greatest lower bound of the radii R of convergence, for which the given polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ constitute a basis in $\mathscr{F}_R$. We denote this infimum by $W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ and derive a new determination of the Whittaker constants $W(\mathscr{D})$ with help of two theorems due to Frank and Buckholtz [3] and Frank and Shaw [7]:

$$\frac{1}{W(\mathscr{D})} = \sup\{W_{\mathscr{D}}((z_k)_{k=0}^{\infty})\}$$

where the supremum is taken over all sequences $(z_k)_{k=0}^{\infty}$ with $|z_k| = d_{k+1}^{-1}$ (see proposition 5).

The Whittaker constant W of the ordinary derivative is defined to be the greatest positive number c with the following property: If, for an entire function f, $\tau(f) = \lim_{n\to\infty} \sup |f^{(n)}(0)|^{1/n} < c$ and each of $f, f', f'',\ldots,$ has a zero in closed disc $|z| = 1$, then $f \equiv 0$.

Let

$$H_n = \max |G_n(0; z_0, \ldots, z_{n-1})|,$$

where the maximum is taken over all sequences $(z_k)_{k=0}^{n-1}$ whose terms lie on $|z| = 1$.

Then

$$W = (\lim_{n\to\infty} H_n^{1/n})^{-1} = (\sup_{1\leqslant n<\infty} H_n^{1/n})^{-1}$$

(see Evgrafov [6], Buckholtz [2]).

For the operator $\mathscr{S}$ define

$$h_n = \max |B_n(0; z_0, \ldots, z_{n-1})|,$$

where the maximum is taken over all sequences $(z_k)_{k=0}^{n-1}$ whose terms lie on $|z| = 1$.

Define

$$W(\mathscr{S}) := (\sup_{1\leqslant n<\infty} h_n^{1/n})^{-1} = (\lim_{n\to\infty} h_n^{1/n})^{-1}$$

(see Buckholtz [1]).

With some restrictive conditions on the sequence $(d_k)_{k=1}^{\infty}$, set in general

$$\mathscr{H}_n = \max |Q_n(0; z_0, \ldots, z_{n-1})|,$$

where the maximum is taken over all sequences $(z_k)_{k=0}^{n-1}$ whose terms lie on $|z| = 1$, and define

$$W(\mathscr{D}) := (\sup_{1\leqslant n<\infty} \mathscr{H}_n^{1/n})^{-1} = (\lim_{n\to\infty} \mathscr{H}_n^{1/n})^{-1}$$

(see Buckholtz and Frank [3]).

$W(\mathscr{D})$ is called the Whittaker constant belonging to the operator $\mathscr{D}$. If we suppose some other restrictive conditions on the sequence $(d_k)_{k=1}^{\infty}$ (see Section 3. formula (3.6)), then we can sharpen proposition 1 to the following result (proposition 4): The polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ constitute a basis in $\mathscr{F}_R$ if and only if for each $r < R$ there exists a number $r' < R$ such that

$$d_1 d_2 \cdots d_n M(r; Q_n) = O(r'^n),$$

where $M(r; Q_n)$ is the maximum modulus.

Finally we give the solution of an interpolation problem with the help of Proposition 4: for each function f holomorphic in the disc $|z| \leqslant W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ there exists a uniquely determined function g holomorphic in the disc $|z| \leqslant W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ such that

$$e_n \mathscr{D}^n g(z_n) = (f^{(n)}(0))/n!$$

for $n = 0, 1, 2, \ldots$ (see Proposition 6).

2. Generalized387 Polynomials and Whittaker Constants

Let $f \in \mathscr{F}_R$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We define the so called Gel'fond–Leont'ev derivative $\mathscr{D}$ by

$$\mathscr{D}f(z) = \sum_{k=1}^{\infty} d_k a_k z^{k-1},$$

where $(d_k)_{k=1}^{\infty}$ is a nondecreasing sequence of positive numbers Let $e_0 = d_0 = 1$ and $e_k = (d_1 d_2 \cdots d_k)^{-1}$ for $k \geqslant 1$; then the successive iterates $\mathscr{D}^m$ of $\mathscr{D}$ can be written as

$$\mathscr{D}^m f(z) = \sum_{k=m}^{\infty} \frac{e_{k-m}}{e_k} a_k z^{k-m} \tag{2.1}$$

(see Gel'fond and Leont'ev [8]).

Now let $(z_k)_{k=0}^{\infty}$ be a sequence of complex numbers and define polynomials $Q_n(z; z_0, z_1, \ldots, z_{n-1})$ by

$$Q_0(z) \equiv 1 \qquad \text{and}$$

$$Q_n(z; z_0, z_1, \ldots, z_{n-1}) = e_n z^n - \sum_{k=0}^{n-1} e_{n-k} z_k^{n-k} Q_k(z; z_0, z_1, \ldots, z_{k-1}) \tag{2.2}$$

for $n = 1, 2, \ldots$. It follows (see Buckholtz and Frank [3])

$$Q_n(\lambda z; \lambda z_0, \lambda z_1, \ldots, \lambda z_{n-1}) = \lambda^n Q_n(z; z_0, \ldots, z_{n-1}), \tag{2.3}$$

where $\lambda \in C$,

$$Q_n(z_0; z_0, \ldots, z_{n-1}) = 0 \qquad n \geqslant 1 \tag{2.4}$$

$$\mathscr{D}^m Q_n(z; z_0, \ldots, z_{n-1}) = Q_{n-m}(z; z_m, \ldots, z_{n-1}), \qquad 0 = m = n \tag{2.5}$$

$$\mathscr{D}^m Q_n(z_m; z_0, \ldots, z_{n-1}) = \delta_{nm}, \tag{2.6}$$

$$Q_n(z; z_0, \ldots, z_{n-1}) = \sum_{k=0}^{n} Q_{n-k}(0; z_k, \ldots, z_{n-1}) e_k z^k. \tag{2.7}$$

Equation (2.6) says that the linear functionals

$$\mathscr{L}_n(f) = \mathscr{D}^n f(z_n) \tag{2.8}$$

are biorthogonal to the polynomials $Q_n(z; z_0, ..., z_{n-1})$. The polynomials Q_n reduce to the Gončarov polynomials if $d_k = k$ and to the remainder polynomials if $d_k \equiv 1$ for $k = 1, 2, ...$.

Now we shall suppose that $(d_{k+1}/d_k)_{k=1}^{\infty}$ is a nonincreasing sequence with limit 1.

An easy consequence of a theorem due to Frank and Shaw ([7], pg. 10) is

THEOREM A. *If* $(z_k)_{k=0}^{\infty}$ *is a sequence of complex numbers such that* $|z_k| \leqslant (W(\mathscr{D})/d_{k+1})\rho$, *then the corresponding polynomials* $Q_n(z; z_0, z_1, ..., z_{n-1})$ *constitute a basis in any space* $\mathscr{F}_R$ $(\rho < R \leqslant \infty)$.

(For the special cases $d_k = k$ and $d_k \equiv 1$ see Dragilev [4] respectively Pommiez [16], Dragilev and Čuhlova [5] and Buckholtz and Frank [3]). That the constant $W(\mathscr{D})$ is best possible in Theorem A is an easy consequence of a theorem due to Buckholtz and Frank [3]:

THEOREM B. *For each* $c > W(\mathscr{D})$ *there exists a sequence* $(z_k^*)_{k=0}^{\infty}$ *with* $|z_k^*| = (c/d_{k+1})\rho$, *such that the corresponding polynomials* $Q_n(z; z_1^*, ..., z_{n-1}^*)$ *do not constitute a basis in a space* $\mathscr{F}_R(\rho < R \leqslant \infty)$.

(For the special case $d_k = k$ see Dragilev [4]).

3. Biorthogonal Sequences and Bases in the Space $\mathscr{F}_R$

$\mathscr{F}_R$ is a nuclear Fréchet space (see Pietsch [15]), the topology of which can be defined by the system of norms

$$\|f\|_r = \max_{|z|=r} |f(z)|, \qquad 0 < r < R, \ \ f \in \mathscr{F}_R .$$

It suffices to take a sequence of norms $\|\cdot\|_{r_n}$ for a sequence $r_n \nearrow R$. In order to introduce a topology on the dual space $\mathscr{F}_R'$ of $\mathscr{F}_R$, one defines a system of unbounded norms by

$$\| L \|_r' = \sup\{| L(f)| : f \in \mathscr{F}_R , \|f\|_r \leqslant 1\}$$

for $L \in \mathscr{F}_R'$.

The topology of the strong dual on $\mathscr{F}_R'$ coincides with the inductive limit topology determined in $\mathscr{F}_R'$ by the system of unbounded norms $\|\cdot\|_r'$ $(0 < r <$

R). A linear functional L on $\mathscr{F}_R$ is continuous if and only if there exists a number $r < R$ such that

$$\| L \|_r' < \infty \qquad \text{(see Rolewicz [17])}.$$

The space $\mathscr{F}_R$ can be consider as a Köthe sequence space (see Köthe [13])

$$\Lambda_R = \left\{ \xi = (\xi_k)_{k=0}^{\infty} : \| \xi \|_r := \sum_{k=0}^{\infty} | \xi_k | \, r^k < \infty \text{ for each } r < R \right\}$$

The isomorphism T between Λ_R and $\mathscr{F}_R$ is given by

$$T((\xi_k)_{k=0}^{\infty}) = \sum_{k=0}^{\infty} \xi_k z^k \in \mathscr{F}_R .$$

(see Rolewicz [17]).

The dual space of Λ_R is again given by a sequence space

$$\Lambda_R' = \left\{ \eta = (\eta_k)_{k=0}^{\infty} : \exists r < R \text{ with } \| \eta \|_r' := \sup_k \frac{| \eta_k |}{r^k} < \infty \right\}$$

The duality is defined by

$$\eta(\xi) = \sum_{k=0}^{\infty} \xi_k \eta_k , \qquad \text{for} \quad \xi \in \Lambda_R \quad \text{and} \quad \eta \in \Lambda_R' .$$

Now let $(f_n, L_n)_{n=0}^{\infty} (f_n \in \mathscr{F}_R, L_n \in \mathscr{F}_R'$ or $f_n = (f_{nk})_{k=0}^{\infty} \in \Lambda_R, L_n = (L_{nk})_{k=0}^{\infty} \in \Lambda_R')$ be a complete biorthogonal sequence for $\mathscr{F}_R$, i.e. the finite linear combinations of the elements f_n are dense in $\mathscr{F}_R$ and $L_n(f_m) = \delta_{nm}$.

THEOREM C. *A complete biorthogonal sequence* $(f_n, L_n)_{n=0}^{\infty}$ *for* $\mathscr{F}_R$ *constitutes a basis in* $\mathscr{F}_R$, *i.e. each* $f \in \mathscr{F}_R$ *has a unique expansion* $f = \sum_{n=0}^{\infty} L_n(f) f_n$, *if and only if one of the following three equivalent conditions is satisfied: for each* r $(0 < r < R)$ *there exists a number* r' $(0 < r' < R)$ *such that*

$$\sup_n \| L_n \|_{r'}' \, \| f_n \|_r < \infty; \tag{3.1}$$

for each $f \in \mathscr{F}_R$ *and for each* r $(0 < r < R)$

$$\sup_n | L_n(f) | \, \| f_n \|_r < \infty; \tag{3.2}$$

for each r $(0 < r < R)$ *there exists a number* r' $(0 < r' < R)$ *such that*

$$\sup_n \left[\left(\sup_k \frac{| L_{nk} |}{r'^k} \right) \sum_{k=0}^{\infty} | f_{nk} | \, r^k \right] < \infty. \tag{3.3}$$

For the proof see [10], for further details and applications see [10, 11, 12].

Now we can characterize those sequences $(z_k)_{k=0}^{\infty}$ for which the corresponding polynomials $Q_n(z; z_0, z_1, ..., z_{n-1})$ constitute a basis in the space $\mathscr{F}_R$.

PROPOSITION 1. *Let $(z_k)_{k=0}^{\infty}$ be a sequence of complex numbers and $(d_k)_{k=1}^{\infty}$ a nondecreasing sequence of positive numbers. Let $d_0 = e_0 = 1$ and $e_k = (d_1 d_2 \cdots d_k)^{-1}$ for $k \geqslant 1$. The corresponding polynomials $Q_n(z; z_0, z_1, ..., z_{n-1})$ constitute a basis in $\mathscr{F}_R$ $(0 < R \leqslant \infty)$, if and only if for each r $(0 < r < R)$ there exists a number r' $(0 < r' < R)$ such that*

$$\sup_n \left[\left(\sup_{m \geqslant n} \frac{e_{m-n}}{e_m r'^m} \mid z_n \mid^{m-n}\right) \sum_{k=0}^{n} \mid Q_{n-k}(0; z_k, ..., z_{n-1}) \mid e_k r^k\right] < \infty \qquad (3.4)$$

Proof. In view of (3.3) we define

$$f_{nk} = Q_{n-k}(0; z_k, ..., z_{n-1})\, e_k \qquad \text{for} \quad k \leqslant n \quad \text{and}$$

$$f_{nk} = 0 \qquad \text{for} \quad k > n;$$

$$L_{nm} = \frac{e_{m-n}}{e_m} z_n^{m-n} \qquad \text{for} \quad m \geqslant n \quad \text{and}$$

$$L_{nm} = 0 \qquad \text{for} \quad m < n.$$

Then

$$\sum_{k=0}^{\infty} f_{nk} L_{nk} = 1, \qquad \sum_{k=0}^{\infty} f_{nk} L_{lk} = 0 \qquad \text{for} \quad l > n, \quad \text{and for} \quad l < n$$

$$\sum_{k=0}^{\infty} f_{nk} L_{lk} = \sum_{k=l}^{n} f_{nk} L_{lk} = \sum_{k=l}^{n} Q_{n-k}(0; z_k, ..., z_{n-1})\, e_k e_k^{-1} e_{k-l} z_l^{k-l}$$

$$= \sum_{j=0}^{n-l} Q_{n-l-j}(0; z_{l+j}, ..., z_{n-1})\, e_j z_l^j = Q_{n-l}(z_l; z_l, ..., z_{n-1}) = 0$$

by (2.7) and (2.4).

This means: $((f_{nk})_{k=0}^{\infty}, (L_{nm})_{m=0}^{\infty})_{n=0}^{\infty}$ is a biorthogonal sequence in Λ_R, which is complete since the canonical basis-elements $(\delta_{nk})_{k=0}^{\infty}$ in Λ_R are representable as finite linear combinations of the sequences $(f_{nk})_{k=0}^{\infty}$. Applying Theorem C, condition (3.3), our proof is finished.

PROPOSITION 2. *Let $(z_k)_{k=0}^{\infty}$, $(d_k)_{k=0}^{\infty}$ and $(e_k)_{k=0}^{\infty}$ be as in proposition 1 and suppose that the corresponding polynomials $Q_n(z; z_0, ..., z_{n-1})$ constitute a basis in a space $\mathscr{F}_R$ $(0 < R < \infty)$. Then they are a basis in any space $\mathscr{F}_{R'}$ $(R' > R)$.*

With the help of proposition 1 the proof is just the same as the proof of proposition 1 in [12]. Now we suppose that the polynomials $Q_n(z; z_0, \dots, z_{n-1})$ constitute a basis in a space $\mathscr{F}_R$ ($R < \infty$) for a given sequence $(z_k)_{k=0}^{\infty}$ and an operator $\mathscr{D}$ and define a constant by

$$W_{\mathscr{D}}((z_k)_{k=0}^{\infty}) = \inf\{R > 0\colon (3\ 4) \text{ is valid for } \mathscr{F}_R\}. \tag{3.5}$$

Then the following result is an immediate consequence of proposition 1 and 2:

PROPOSITION 3. *Any function f holomorphic in a disc $|z| < R$, where $R > W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$, is uniquely representable in a series*

$$f(z) = \sum_{n=0}^{\infty} \mathscr{D}^n f(z_n)\, Q_n(z; z_0, \dots, z_{n-1})$$

and $\mathscr{D}^n f(z_n) = 0$ for $n = 0, 1, 2, \dots$, implies $f \equiv 0$. If $W_{\mathscr{D}}((z_k)_{k=0}^{\infty}) > 0$, then there exists a function f holomorphic in a disc $|z| < R^*$, where $0 < R^* < W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$, which is not representable in a series of the polynomials $Q_n(z; z_0, \dots, z_{n-1})$.

These results can be improved, if one assumes some restrictive conditions on the sequences $(z_k)_{k=0}^{\infty}$ and $(d_k)_{k=0}^{\infty}$.

PROPOSITION 4. *Let $(d_k)_{k=1}^{\infty}$ be a sequence of positive numbers such that*

$$d_1 d_{n+k} \leqslant d_{n+1} d_k \tag{3.6}$$

for each $n, k = 1, 2, \dots$, $d_0 = 1$ and $(z_k)_{k=0}^{\infty}$ be a sequence of complex numbers with $|z_k| = d_{k+1}^{-1}$ for $k = 0, 1, 2, \dots$.

Then the corresponding polynomials $Q_n(z; z_0, \dots, z_{n-1})$ constitute a basis in $\mathscr{F}_R$ ($0 < R \leqslant \infty$) if and only if for each r ($0 < r < R$) there exists a number r' ($0 < r' < R$) such that

$$e_n^{-1} \| Q_n \|_r = O(r'^n). \tag{3.7}$$

And

$$W_{\mathscr{D}}((z_k)_{k=0}^{\infty}) = \inf\{R > 0\colon \forall r < R\ \exists r' < R \text{ with } e_n^{-1} \| Q_n \|_r = O(r'^n)\}. \tag{3.8}$$

Proof. In view of (3.4) we have to compute

$$\sup_{m \geqslant n} \frac{e_{m-n}}{e_n r'^m} |z_n|^{m-n} = \sup_{m \geqslant n} \frac{e_{m-n} d_{n+1}^{n-m}}{e_m r'^m} \tag{3.9}$$

Here we can assume that $r' \geqslant d_1^{-1}$, because for $n = 0$ one sees that (3.9) is infinite for $r' < d_1^{-1}$, and by (3.4) the polynomials $Q_n(z; z_0, ..., z_{n-1})$ could not constitute a basis in a space $\mathscr{F}_R$ for $R \leqslant d_1^{-1}$.

With the assumption that $r' \geqslant d_1^{-1}$ we can verify that

$$\sup_{m \geqslant n} \frac{e_{m-n} d_{n+1}^{n-m}}{e_m r'^m} = e_n^{-1} r'^n. \tag{3.10}$$

Let $k \geqslant 0$; by our assumption on the sequence $(d_k)_{k=0}^{\infty}$ we have

$$d_{n+1} d_{n+2} \cdots d_{n+k} d_{n+1}^{-k} \leqslant d_1 d_2 \cdots d_k d_1^{-k},$$

which implies

$$d_{n+1} d_{n+2} \cdots d_{n+k} d_{n+1}^{-k} \leqslant d_1 d_2 \cdots d_k r'^k;$$

thus we obtain

$$d_1 d_2 \cdots d_{n+k} (d_1 d_2 \cdots d_k)^{-1} d_{n+1}^{-k} \leqslant d_1 d_2 \cdots d_n r'^k,$$

and if we divide both sides by r'^{n+k}, we get

$$\frac{e_k d_{n+1}^{-k}}{e_{n+k} r'^{n+k}} \leqslant e_n^{-1} r'^{-n}.$$

Set $m = n + k$ and observe that, for $m = n$, $e_{m-n}\, d_{n+1}^{n-m} / e_m r'^m$ equals $e_n^{-1} r'^{-n}$, which proves (3.10). For each $f \in \mathscr{F}_R$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$ the two systems of norms

$$(\|f\|_r\,,\ 0 < r < R) \qquad \text{and} \qquad \left(\sum_{k=0}^{\infty} |\, a_k \,|\, r^k,\ 0 < r < R\right)$$

are equivalent, i.e. for each $r < R$ there exists a number $r' < R$ and a constant K_1 such that

$$\|f\|_r \leqslant K_1 \sum_{k=0}^{\infty} |\, a_k \,|\, r'^k \qquad \text{for each} \quad f \in \mathscr{F}_R$$

and for each $r < R$ there exists a number $r' < R$ and a constant K_2 such that

$$\sum_{k=0}^{\infty} |\, a_k \,|\, r^k \leqslant K_2 \|f\|_{r'} \qquad \text{for each} \quad f \in \mathscr{F}_R$$

(the last inequality follows by Cauchy's integral formulas for the derivatives of a holomorphic function). Now (3.7) is equivalent to (3.4) and (3.8) is just the same as (3.5). Q.E.D.

Condition (3.6) is satisfied for $d_k = k$ $(k = 1, 2,...)$ or $d_k \equiv 1$. With another assumption on the sequence $(d_k)_{k=1}^{\infty}$ one obtains a new characterization of the Whittaker constants $W(\mathscr{D})$:

PROPOSITION 5. *Let $(d_k)_{k=1}^{\infty}$ denote a nondecreasing sequence of positive numbers such that the sequence $(d_{k+1}, d_k^{-1})_{k=1}^{\infty}$ is nonincreasing and has limit* 1. *Then*

$$\frac{1}{W(\mathscr{D})} = \sup\{W_{\mathscr{D}}((z_k)_{k=0}^{\infty})\}, \tag{3.11}$$

where the supremum is taken over all sequences $(z_k)_{k=0}^{\infty}$ with $|z_k| = d_{k+1}^{-1}$.

This follows by the definition of the constant $W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ and by theorem A and B setting $\rho = 1/W(\mathscr{D})$.

4. INTERPOLATION

As an application of the propositions in part 3 we obtain the following result

PROPOSITION 6. *Let $(d_k)_{k=1}^{\infty}$ be a sequence of positive numbers satisfying* (3.6) *and $(z_k)_{k=0}^{\infty}$ be a sequence of complex numbers with $|z_k| = d_{k+1}^{-1}$. Then for each sequence $(a_k)_{k=0}^{\infty}$ of complex numbers such that*

$$\limsup_{n\to\infty} |a_n|^{1/n} < [W_{\mathscr{D}}((z_k)_{k=0}^{\infty})]^{-1}$$

there exists an uniquely determined sequence $(c_m)_{m=0}^{\infty}$ of complex numbers with

$$\limsup_{m\to\infty} |c_m|^{1/m} < [W_{\mathscr{D}}((z_k)_{k=0}^{\infty})]^{-1}$$

such that

$$e_n \sum_{m=n}^{\infty} \frac{e_{m-n}}{e_m} c_m z_n^{m-n} = a_n \qquad \text{for} \quad n = 0, 1, 2,\ldots . \tag{4.1}$$

In other words: for each function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in the disc $|z| \leqslant W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ there exists a uniquely determined function $g(z) = \sum_{m=0}^{\infty} c_m z^m$ holomorphic in the disc $|z| \leqslant W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ such that

$$e_n \mathscr{D}^n g(z_n) = (f^{(n)}(0))/n! \qquad \textit{for} \quad n = 0, 1, 2,\ldots . \tag{4.2}$$

Proof. In view of Theorem A the constant $W_{\mathscr{D}}((z_k)_{k=0}^{\infty})$ exists and satisfies $W_{\mathscr{D}}((z_k)_{k=0}^{\infty}) < \infty$.

Choose a number $R > 0$ such that $\limsup |a_n|^{1/n} \leqslant 1/R < [W_{\mathscr{D}}((z_k)_{k=0}^{\infty})]^{-1}$, then by Proposition 3 and Proposition 4 for each $r < R$ there exists a number $r' < R$ and a constant K such that

$$e_n^{-1} \| Q_n \|_r \leqslant K r'^n.$$

Now an $\epsilon > 0$ such that

$$r' \left(\frac{1}{R} + \epsilon \right) \leqslant 1;$$

for this $\epsilon > 0$ there exists a number $N_\epsilon > 0$ with

$$| a_n | \leqslant \left(\frac{1}{R} + \epsilon \right)^n \qquad \text{for each} \quad n \geqslant N_\epsilon .$$

It follows that

$$| a_n | \, e_n^{-1} \| Q_n \|_r \leqslant K \left(\frac{1}{R} + \epsilon \right)^n r'^n \leqslant K$$

for each $n \geqslant N_\epsilon$.

This implies

$$\sup_{0 \leqslant n < \infty} | a_n | \, e_n^{-1} \| Q_n \|_r < \infty \tag{4.3}$$

for each $r < R$. (Compare (3.2)).

The polynomials $Q_n(z; z_0, \ldots, z_{n-1})$ constitute a basis in $\mathscr{F}_R$; now by the theorem of Dynin–Mitiagin on the absoluteness of a basis in a nuclear Fréchet space (see Mitiagin [14] or Pietsch [15]) we have that for each $r < R$ there exists a number $r' < R$ such that

$$\sum_{n=0}^{\infty} \frac{\| Q_n \|_r}{\| Q_n \|_{r'}} < \infty \tag{4.4}$$

(see Rolewicz [17] pg. 189).

Collecting the last results we obtain

$$\sum_{n=0}^{\infty} | a_n | \, e_n^{-1} \| Q_n \|_r = \sum_{n=0}^{\infty} | a_n | \, e_n^{-1} \| Q_n \|_{r'} \frac{\| Q_n \|_r}{\| Q_n \|_{r'}}$$

$$\leqslant \sup_{0 \leqslant n < \infty} | a_n | \, e_n^{-1} \| Q_n \|_{r'} \sum_{n=0}^{\infty} \frac{\| Q_n \|_r}{\| Q_n \|_{r'}} < \infty$$

Since $\mathscr{F}_R$ is a complete space and the polynomials $Q_n(z; z_0, ..., z_{n-1})$ constitute a basis in $\mathscr{F}_R$, there exists a function $g \in \mathscr{F}_R$ with

$$\mathscr{D}^n g(z_n) = a_n e_n^{-1}$$

and Proposition 6 is proved.

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